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★**Derived category methods in commutative algebra.**

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Homological ideas grew out of attempts to investigate deeper properties of topological spaces by means of studying certain algebraic objects, namely homology and cohomology groups. These ideas took abstract forms later and planted the early seeds of homological algebra in the framework of category theory. At its core, homological algebra seeks to construct and study derived functors, which measure the extent to which a functor fails to be exact. The theory of derived categories was developed by Grothendieck and his school to provide a framework that can be deployed to assemble all derived functors of a given functor into a single derived functor, and further study them in a coherent and flexible way.

In commutative algebra, derived category methods have proven to be highly efficient and indispensable. Homological invariants and homological dimensions are extensively used in commutative algebra to provide deeper insight into the structure of rings and modules. The book under review offers a clear and comprehensive exposition of these methods, carefully guiding the reader from classical constructions to advanced techniques. The text is structured into three parts—Foundations, Tools, and Applications—so that readers first acquire the necessary categorical and homological machinery, then develop the technical tools required for computation, and finally see how these tools can be applied to deep questions in commutative algebra. Throughout, the authors emphasize explicit constructions, illustrative examples, and useful exercises, making advanced concepts accessible to graduate students while maintaining the rigor expected by researchers.

Chapter 1 introduces the category of modules over a ring, including standard isomorphisms, exact functors, and evaluation homomorphisms. This chapter establishes the essential language and tools needed for later developments.

Chapters 2–4 systematically study the theory of complexes and their homologies. The homomorphism functor as well as the tensor product functor are explored in detail. Moreover, foundational categorical constructions such as (co)limits, filtered colimits, and towers are developed in the context of complexes.

Resolutions are fundamental in homological algebra. They are essential in computing derived functors as well as defining homological dimensions. Chapter 5 presents semi-free, semi-projective, semi-injective, and semi-flat resolutions in depth and rigorously explores their properties.

Chapter 6 constructs the homotopy category and establishes its triangulation. The derived category of a ring is then constructed by means of roof diagrams in the homotopy category, and their properties are justified by using resolutions. Finally, the triangulation of the derived category is worked out completely.

Chapter 7 studies derived functors over the homotopy category and then extends them to the derived category. The two core examples, the derived homomorphism functor and the derived tensor functor, are investigated with great care. Furthermore, the standard isomorphisms regarding these derived functors are probed.

Chapters 8 and 9 offer a thorough treatment of projective, injective, and flat dimensions, along with their Gorenstein counterparts. Inclusion of Gorenstein homological dimensions is particularly valuable, as these topics have gained prominence in recent research yet are rarely presented with such clarity and comprehensiveness.

Chapter 10 develops the theory of dualizing complexes and Grothendieck duality. Moreover, Morita equivalence and the Foxby-Sharp equivalence are treated carefully. Finally, Auslander and Bass categories and their role in the study of Gorenstein dimensions are demonstrated.

Chapter 11 sets forth the theory of torsion and completion functors and their derived functors. This can be perceived as the derived category extension of the theory of local homology and cohomology of modules. The Koszul and Čech complexes and their relations with this theory are treated with precision. The chapter connects these derived constructions to classical commutative algebra results, illustrating the advantages of the derived perspective.

Chapters 12–14 review the essential results in commutative noetherian rings and extend classical notions, such as Krull dimension, depth, and support, into the derived category context. The examples and exercises clarify subtle points, making these chapters particularly useful for readers bridging classical and derived methods. The celebrated Greenlees-May equivalence is explored in detail in Chapter 13.

Support and cosupport theories in the context of commutative rings are developed in great detail in chapters 15–17. Moreover, homological invariants and homological dimensions over commutative rings are studied extensively. Applications to Cohen-Macaulayness, Gorenstein rings, and rigidity of Ext and Tor are illustrated in detail.

The final chapters 18–20 present cornerstone results: Matlis duality, Grothendieck duality, local duality, the New Intersection Theorem, Iversen's amplitude inequality, and a more detailed treatment of Gorenstein dimensions, Gorenstein rings, and regular rings. By framing these classical theorems in the derived category language, the authors reveal both conceptual unity and technical advantage.

The exposition of the book is straightforward and to the point. After a short summary of the main ideas, most sections proceed swiftly to the relevant definitions, examples, and theorems. The proofs are solid, rigorous, and thorough, presenting the ideas in a transparent easy-to-grasp manner. Explicit constructions are preferred over abstract axiomatic approaches, making the text both approachable and rigorous. The authors maintain a good balance between theory and practice. Numerous exercises are offered at the end of each section, allowing the reader to gain hands-on experience with the material.

This book is a landmark in homological methods for commutative algebra. It bridges classical and modern techniques, offering both a comprehensive reference and a pedagogically rich learning resource. The text contains all the results pertaining to the derived category applications in commutative algebra, ranging from classical theorems and constructions to advanced results that can only be found in research papers. The organization of the material and the level of precision employed in the statement of the theorems are outstanding. Reading this book profoundly enhanced my own understanding of derived categories, derived functors, homological dimensions, homological invariants, and their applications, providing tools directly applicable to research.

Derived category methods in commutative algebra is a masterful achievement: rigorous, accessible, and pedagogically superb. It should be read by graduate students seeking a solid foundation in derived categories and by researchers requiring a comprehensive reference. Christensen, Foxby and Holm have not only taught the subject but conveyed its elegance, depth, and utility in a way few other texts have managed. All in all, I identify it as a well-written, readable book that algebra practitioners will enjoy having on their shelves.

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